Weak value analogue in classical stochastic process

Hiroyuki Tomita

Research Center for Quantum Computing,
Kinki University,
Kowakae 3-4-1, Higashi-Osaka, 577-8502, Japan

Abstract

The time evolution of the two-time conditional probability of the classical stochastic process is described in an analogous form of the quantum mechanical wave equations. By using it, we emulate the same strange behaviors as those of the weak value in the quantum mechanics. A negative probability and abnormal expectations of some quantities remarkablely larger than their inherent norms are found in an example of a stochastic Ising spin system.

Keywords: Weak value, Stochastic process, Two-time conditional probability, Stochastic Ising model

1. Introduction

A notion of the weak value proposed by Aharonov et al [1] has brought a new understanding on the quantum observation, i.e. a weak measurement [2] which hardly disturbs the quantum state. The reason of this strange nature of the quantum measurement is that the weak value is defined as an expectation with the condition of two-time observations of the initial and the final states which differ from one another. This condition is very rare case with little probability and is far from the main behavior of a given quantum system. Then the observation of the weak value does not disturb the quantum system not so fatally. As a result of this rather fictitious probability, the weak value happens to be abnormally enhanced from its inherent norm.

The purpose of this letter is to make the mechanism of this abnormal behavior clearer by using a classical stochastic model, in which we can avoid the ambiguity of the complex probability in the quantum case [3].

Email address: tomita@alice.math.kindai.ac.jp (Hiroyuki Tomita)

We introduce a conventional transformation of the stochastic master equation to a self-adjoint form in the following section. A good analogy with the quantum mechanics is found by applying this tarnsformation to the two-time conditional probability (TTCP). This is shown in Sec.3. An example of the stochastic Ising model which shows an abnormal enhancement of the expectations of some quantities with respect to TTCP is given in Sec.4. In Sec.5 we discuss an extention of TTCP to a density matrix form to complete the analogy with the quantum mechanics. The last section is devoted to brief summary and discussions.

2. Self-adjoint form of stochastic master equation

First let us survey the well-known transformation [4] to a self-adjoint form of the stochastic master equation.

Let $\boldsymbol{x}$ be a set of stochastic variables described by a time-dependent conditional probability, $P(\boldsymbol{x}, t|\boldsymbol{x}_{\rm i}, t_{\rm i})$ for $t \geq t_{\rm i}$, which obeys the following stationary, Markoffian master equation, i.e. the Chapman-Kolmogorov *forward* equation,

$$\begin{aligned} \frac{\partial}{\partial t} P(\boldsymbol{x}, t|\boldsymbol{x}_{\rm i}, t_{\rm i}) &= -\sum_{\boldsymbol{x}'} W(\boldsymbol{x} \to \boldsymbol{x}') P(\boldsymbol{x}, t|\boldsymbol{x}_{\rm i}, t_{\rm i}) + \sum_{\boldsymbol{x}'} W(\boldsymbol{x}' \to \boldsymbol{x}) P(\boldsymbol{x}', t|\boldsymbol{x}_{\rm i}, t_{\rm i}) \\ &= -\sum_{\boldsymbol{x}'} L(\boldsymbol{x}, \boldsymbol{x}') P(\boldsymbol{x}', t|\boldsymbol{x}_{\rm i}, t_{\rm i}), \end{aligned} \tag{1}$$

where

$$L(\boldsymbol{x}, \boldsymbol{x}') = \left[\sum_{\boldsymbol{x}''} W(\boldsymbol{x} \to \boldsymbol{x}'') \right] \delta(\boldsymbol{x} - \boldsymbol{x}') - W(\boldsymbol{x}' \to \boldsymbol{x}).$$

The matrix L has an eigenvalue $\lambda_0 = 0$ correspondding to the steady state,

$$P_0(\boldsymbol{x}) = \lim_{t - t_{\rm i} \to \infty} P(\boldsymbol{x}, t|\boldsymbol{x}_{\rm i}, t_{\rm i}).$$

Let us introduce a *wave function* related to this forward conditional probability by

$$\psi(\boldsymbol{x}, t|\boldsymbol{x}_{\rm i}, t_{\rm i}) = \phi_0(\boldsymbol{x})^{-1} P(\boldsymbol{x}, t|\boldsymbol{x}_{\rm i}, t_{\rm i}), \quad (t \geq t_{\rm i}) \tag{2}$$

where $\phi_0(\boldsymbol{x}) = P_0(\boldsymbol{x})^{1/2}$. This function ψ obeys the forward wave equation,

$$\frac{\partial}{\partial t} \psi(\boldsymbol{x}, t) = -\sum_{\boldsymbol{x}'} H(\boldsymbol{x}, \boldsymbol{x}') \psi(\boldsymbol{x}', t), \tag{3}$$

where H is defined by

$$H(\boldsymbol{x}, \boldsymbol{x'}) = \phi_0(\boldsymbol{x})^{-1} L(\boldsymbol{x}, \boldsymbol{x'}) \phi_0(\boldsymbol{x'}). \tag{4}$$

For the time being the initial condition $(\boldsymbol{x}_{\mathrm{i}}, t_{\mathrm{i}})$ in ψ is abbreviated. The function $\phi_0(\boldsymbol{x})$ is an eigenfunction of Eq.(3) for $\lambda_0 = 0$.

The merit of this transformation is that the eigenvalue problem of a given master equation is simplified, if the matrix H is symmetric, i.e.

$$H(\boldsymbol{x}, \boldsymbol{x'}) = H(\boldsymbol{x'}, \boldsymbol{x}).$$

This situation is widely expected when the detailed balance codition, i.e. the time-reversal symmetry [5],

$$P_0(\boldsymbol{x}) W(\boldsymbol{x} \to \boldsymbol{x'}) = P_0(\boldsymbol{x'}) W(\boldsymbol{x'} \to \boldsymbol{x}),$$

or equivalently,

$$L(\boldsymbol{x}, \boldsymbol{x'}) P_0(\boldsymbol{x'}) = L(\boldsymbol{x'}, \boldsymbol{x}) P_0(\boldsymbol{x}), \tag{5}$$

is satisfied. In this case the eigenvalues of H are all real, and non-negative, if the steady state is stable. Therefore, $\phi_0(\boldsymbol{x})$ is the ground state.

A useful example is the Fokker-Planck equation for a single, continuous stochastic variable x,

$$\frac{\partial}{\partial t} P(x,t) = -\mathcal{L}[x] P(x,t), \ \mathcal{L}[x] = -\frac{\partial}{\partial x} \left(F'(x) + \frac{\epsilon}{2} \frac{\partial}{\partial x} \right), \tag{6}$$

which describes a one-dimensional Brownian motion in a potential $F(x)$ with a small diffusion constant ϵ. By using its steady state solutions,

$$P_0(x) \propto \exp\left[-2F(x)/\epsilon\right] \quad \text{and} \quad \phi_0(x) \propto \exp\left[-F(x)/\epsilon\right],$$

we find the continuous variable version of the above formulations,

$$\mathcal{H}[x] = \frac{1}{\epsilon} \left[-\frac{\epsilon^2}{2} \frac{\partial^2}{\partial x^2} + V(x) \right], \quad V(x) = \frac{1}{2} \left[F'(x)^2 - \epsilon F''(x) \right]. \tag{7}$$

Thus the Fokker-Planck equation is transformed into a self-adjoint form of an imaginary-time Schrödinger equation,

$$-\epsilon \frac{\partial}{\partial t} \psi(x,t) = \left[-\frac{\epsilon^2}{2} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x,t),$$

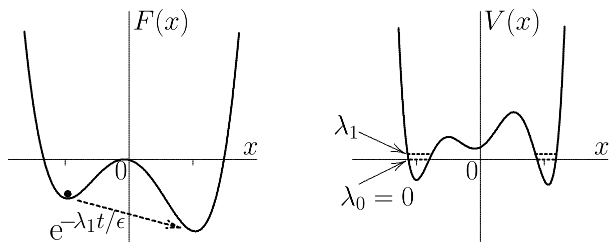

Figure 1: Stochastic decay process of the metastable state.

and its eigenvalue problem results in a familiar one of the quantum mechanics.

Figure.1 shows an early application [6] to the so-called Kramers escape problem [7]. The stochastic decay rate of the metastable state in a double-well potential $F(x)$ is given by the first excited eigenvalue λ_1 of the corresponding Schrödinger potential $V(x)$. The first excited state is almost degenerated with the ground state for the small diffusion constant ϵ.

3. Two-time conditional probability

So far the quantum mechanical reformulation merely helps us to simplify the eigenvalue problem of a given master equation. None of the remarkable quantum mechanical phenomena appears, until we are concerned with the TTCP,

$$P(\boldsymbol{x}, t|\boldsymbol{x}_\mathrm{i}, t_\mathrm{i}; \boldsymbol{x}_\mathrm{f}, t_\mathrm{f}), \quad t_\mathrm{i} \leq t \leq t_\mathrm{f} \ . \quad (\ ;\ \text{denoting ‘and’}) \tag{8}$$

By using the Markoffian property and the well-known equality of the simplest Bayes' theorem,

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A),$$

repeatedly, the TTCP can be written in the following form with a pair of the wave functions as

$$P(\boldsymbol{x}, t|\boldsymbol{x}_\mathrm{i}, t_\mathrm{i}; \boldsymbol{x}_\mathrm{f}, t_\mathrm{f}) = \frac{1}{\langle \psi_\mathrm{f} | \psi_\mathrm{i} \rangle} \overline{\psi}(\boldsymbol{x}, t|\boldsymbol{x}_\mathrm{f}, t_\mathrm{f}) \psi(\boldsymbol{x}, t|\boldsymbol{x}_\mathrm{i}, t_\mathrm{i}), \tag{9}$$

where the conjugate wave function $\overline{\psi}$ is related to the so-called *posterior* conditional probability, $\overline{P}(\boldsymbol{x}, t|\boldsymbol{x}_\mathrm{f}, t_\mathrm{f})$ for $t \leq t_\mathrm{f}$, by

$$\overline{\psi}(\boldsymbol{x}, t|\boldsymbol{x}_\mathrm{f}, t_\mathrm{f}) = \phi_0(\boldsymbol{x})^{-1} \overline{P}(\boldsymbol{x}, t|\boldsymbol{x}_\mathrm{f}, t_\mathrm{f}), \tag{10}$$

and obeys the *backward* wave equation,

$$\frac{\partial}{\partial t}\overline{\psi}(\boldsymbol{x},t)=\sum_{\boldsymbol{x'}}H^{\dagger}(\boldsymbol{x},\boldsymbol{x'})\overline{\psi}(\boldsymbol{x'},t). \tag{11}$$

Here $H^{\dagger}$ is the hermite conjugate of H, i.e. the transposed matrix in the present case. The eigensystem is common with the forward equation Eq.(3), when H is hermite, i.e. real and symmetric as has been assumed here.

The denominator in Eq.(9) is the weight of overlap between the two wave functions defined by an inner product,

$$\langle\psi_{\rm f}|\psi_{\rm i}\rangle=\sum_{\boldsymbol{x'}}\overline{\psi}(\boldsymbol{x'},t|\boldsymbol{x}_{\rm f},t_{\rm f})\psi(\boldsymbol{x'},t|\boldsymbol{x}_{\rm i},t_{\rm i}). \tag{12}$$

Of course this quantity is real, while the corresponding quantity in the quantum mechanics is complex in general.

Let us define the ket- and the bra-vectors by

$$|\psi_{\rm i}(t)\rangle=\{\psi(\boldsymbol{x},t|\boldsymbol{x}_{\rm i},t_{\rm i})\}^{\dagger}\ \text{and}\ \langle\psi_{\rm f}(t)|=\{\overline{\psi}(\boldsymbol{x},t|\boldsymbol{x}_{\rm f},t_{\rm f})\}.$$

Then the wave equations Eqs.(3) and (11) are rewritten in the quantum mechanical form as

$$\frac{\partial}{\partial t}|\psi_{\rm i}(t)\rangle=-H|\psi_{\rm i}(t)\rangle\ \text{ and }\ \frac{\partial}{\partial t}\langle\psi_{\rm f}(t)|=\langle\psi_{\rm f}(t)|H. \tag{13}$$

Henceforth, H is called the Hamitonian.

By using this pair of the Schrödinger equations it is shown that the overlap integral, $\langle\psi_{\rm f}|\psi_{\rm i}\rangle$ given by Eq.(12) does not depend on the current time t, i.e.

$$\frac{\partial}{\partial t}\langle\psi_{\rm f}|\psi_{\rm i}\rangle=\langle\psi_{\rm f}(t)|H|\psi_{\rm i}(t)\rangle-\langle\psi_{\rm f}(t)|H|\psi_{\rm i}(t)\rangle=0.$$

Further it can be shown that this overlap integral has the following properties in the two limits;

$$\begin{cases}\text{(i)}\ \lim\limits_{t_{\rm f}-t_{\rm i}\to\infty}\langle\psi_{\rm f}|\psi_{\rm i}\rangle=1,\\ \\ \text{(ii)}\ \lim\limits_{t_{\rm f}-t_{\rm i}\to 0}\langle\psi_{\rm f}|\psi_{\rm i}\rangle=[\phi_0(\boldsymbol{x}_{\rm f})\phi_0(\boldsymbol{x}_{\rm i})]^{-1}\delta(\boldsymbol{x}_{\rm f}-\boldsymbol{x}_{\rm i}).\end{cases} \tag{14}$$

Note that the two-time conditional expectation (TTCE) of a physical quantity Q with respect to TTCP defined by

$$\begin{aligned}\langle Q\rangle_{(\mathrm{i};\mathrm{f})} &= \sum_{\boldsymbol{x}'} Q(\boldsymbol{x}')P(\boldsymbol{x}',t|\boldsymbol{x}_\mathrm{i},t_\mathrm{i};\boldsymbol{x}_\mathrm{f},t_\mathrm{f}) \\ &= \frac{\langle\psi_\mathrm{f}(t)|Q|\psi_\mathrm{i}(t)\rangle}{\langle\psi_\mathrm{f}|\psi_\mathrm{i}\rangle}, \end{aligned} \tag{15}$$

has just the analogous form of the weak value in the quantum mechanics [3].

Thus the TTCP is a non-linear quantity composed of a product of a pair of the forward and the backward wave functions, and cannot be described by a closed, linear evolution equation. Then it happens that the principle of the probability superposition is violated and the interference of wave functions may occur. However, its example is omitted here because none of nontrivial phenomenon from this view point has been found, yet. The reason may be that the wave functions are always real and possitive in the present case. Let us discuss only the weak value in the rest.

4. Stochastic model of classical Ising spins

An example is a pair of the classical Ising spin $\sigma = \pm 1$ having an exchange interaction,

$$E(\boldsymbol{x}) = -J\sigma_1\sigma_2,$$

where $\boldsymbol{x} = (\sigma_1, \sigma_2)$. Let us number the stochastic variable $\boldsymbol{x}$ in the order, $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$ and choose the following transition matrices,

$$W = \begin{pmatrix} 0 & 1 & 1 & 0 \\ p^2 & 0 & 0 & p^2 \\ p^2 & 0 & 0 & p^2 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{or} \quad L = \begin{pmatrix} 2p^2 & -1 & -1 & 0 \\ -p^2 & 2 & 0 & -p^2 \\ -p^2 & 0 & 2 & -p^2 \\ 0 & -1 & -1 & 2p^2 \end{pmatrix}, \tag{16}$$

where $p = \mathrm{e}^{-\beta J}$. Evidently this transition matrix W satisfies the detailed balance condition,

$$\mathrm{e}^{-\beta E(\boldsymbol{x})}\, W(\boldsymbol{x}\to\boldsymbol{x}') = \mathrm{e}^{-\beta E(\boldsymbol{x}')}\, W(\boldsymbol{x}'\to\boldsymbol{x}),$$

at the steady state, i.e. the thermal equilibrium of a temperature parameter, $\beta = 1/k_\mathrm{B}T$. With use of the equilibrium distribution function,

$$P_0(\boldsymbol{x}) = \frac{1}{2(1+p^2)}\,(1,\ p^2,\ p^2,\ 1) \ \text{ and } \ \phi_0(\boldsymbol{x}) = \frac{1}{\sqrt{2(1+p^2)}}\,(1,\ p,\ p,\ 1),$$

we find the corresponding hermite Hamiltonian,

$$\begin{aligned} H &= \begin{pmatrix} 2p^2 & -p & -p & 0 \\ -p & 2 & 0 & -p \\ -p & 0 & 2 & -p \\ 0 & -p & -p & 2p^2 \end{pmatrix} \\ &= (1+p^2)\ \sigma_0 \otimes \sigma_0 - (1-p^2)\ \sigma_z \otimes \sigma_z - p\ (\sigma_0 \otimes \sigma_x + \sigma_x \otimes \sigma_0), \end{aligned} \tag{17}$$

where σ_x and σ_z are the usual Pauli matrices and σ_0 denotes the two dimensional unit matrix I_2. This is the Hamiltonian of a pair of *quantum* Ising spins with the exchange interaction in a transverse magnetic field.

The eigenvalues and the eigenstates of this Hamiltonian H,

$$\begin{cases} \lambda_0 = 0,\ \lambda_1 = 2p^2,\ \lambda_2 = 2,\ \lambda_3 = 2(1+p^2), \\ \\ |0\rangle = \dfrac{1}{\sqrt{2(1+p^2)}} [\ |\uparrow\uparrow\rangle\ +\ p\ |\uparrow\downarrow\rangle +\ p\ |\downarrow\uparrow\rangle\ +\ |\downarrow\downarrow\rangle\ ], \\ |1\rangle = \dfrac{1}{\sqrt{2}} [\ |\uparrow\uparrow\rangle\ -\ |\downarrow\downarrow\rangle\ ], \\ |2\rangle = \dfrac{1}{\sqrt{2}} [\ |\uparrow\downarrow\rangle\ -\ |\downarrow\uparrow\rangle\ ], \\ |3\rangle = \dfrac{1}{\sqrt{2(1+p^2)}} [\ p\ |\uparrow\uparrow\rangle\ -\ |\uparrow\downarrow\rangle -\ |\downarrow\uparrow\rangle\ +\ p\ |\downarrow\downarrow\rangle\ ], \end{cases}$$

can be easily obtained, where $|0\rangle = |\phi_0\rangle$, the ground state. Here the familiar notations $\uparrow, \downarrow$ are used for $\sigma = \pm 1$. Note that the first excited state is almost degenerated with the ground state for a small transition probability p^2.

By using this eigensystem we can calculate the state vectors, $|\psi_{\mathrm{i}}(t)\rangle$ and $\langle\psi_{\mathrm{f}}(t)|$ for arbitrary initial and final states in just the same manner of the elementary quantum mechanics except for the fact that the time t is imaginary.

An interesting example is the case where the initial and the final states differ from each other, just like the case of the weak value. For example, let

$$\boldsymbol{x}_{\mathrm{i}} = (\uparrow\uparrow) \text{ at } t = 0 \quad \text{and} \quad \boldsymbol{x}_{\mathrm{f}} = (\downarrow\downarrow) \text{ at } t = t_{\mathrm{f}},$$

that is,

$$P(\boldsymbol{x}, 0) = (1, 0, 0, 0) \quad \text{and} \quad \overline{P}(\boldsymbol{x}, t_{\mathrm{f}}) = (0, 0, 0, 1),$$

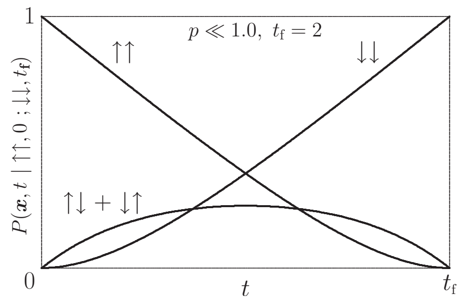

Figure 2: Two-time conditional probability

or equivalently,

$$|\psi_{\rm i}(0)\rangle = \sqrt{2(1+p^2)}\ |{\uparrow\uparrow}\rangle \ \text{ and } \ \langle\psi_{\rm f}(t_{\rm f})| = \sqrt{2(1+p^2)}\ \langle{\downarrow\downarrow}|.$$

By using the eigenvector expansion we obtain,

$$\begin{aligned} |\psi_{\rm i}(t)\rangle &= |0\rangle + \sqrt{1+p^2}\ {\rm e}^{-\lambda_1 t}\ |1\rangle + p\ {\rm e}^{-\lambda_3 t}\ |3\rangle, \\ \langle\psi_{\rm f}(t)| &= \langle 0| - \sqrt{1+p^2}\ {\rm e}^{-\lambda_1 (t_{\rm f}-t)}\langle 1| + p\ {\rm e}^{-\lambda_3 (t_{\rm f}-t)}\langle 3|, \end{aligned} \tag{18}$$

and

$$\langle\psi_{\rm f}|\psi_{\rm i}\rangle = 1 - (1+p^2)\ {\rm e}^{-\lambda_1 t_{\rm f}} + p^2\ {\rm e}^{-\lambda_3 t_{\rm f}}\ (>0). \tag{19}$$

The TTCP is shown in Figure.2. This result itself is very natural and well-expected, all probabilities being always non-negative.

A strange behavior appears when we use the basis $\{|k\rangle,\ k=0,1,2,3\}$, the eigenstates of the Hamiltonian H instead of the spin states $\{|\boldsymbol{x}\rangle = |\sigma_1\sigma_2\rangle\}$. We can calculate the probability, i.e. the TTCE of the projection operator $|k\rangle\langle k|$ onto each eigenstate k in the same manner. The result is given by

$$\begin{aligned} P(0,t) &= \frac{\langle\psi_{\rm f}(t)|0\rangle\langle 0|\psi_{\rm i}(t)\rangle}{\langle\psi_{\rm f}|\psi_{\rm i}\rangle} &&= \frac{1}{\langle\psi_{\rm f}|\psi_{\rm i}\rangle}\ , \\ P(1,t) &= \frac{\langle\psi_{\rm f}(t)|1\rangle\langle 1|\psi_{\rm i}(t)\rangle}{\langle\psi_{\rm f}|\psi_{\rm i}\rangle} &&= -\frac{(1+p^2){\rm e}^{-\lambda_1 t_{\rm f}}}{\langle\psi_{\rm f}|\psi_{\rm i}\rangle}\ \ (<0\ )\ , \\ P(2,t) &= \frac{\langle\psi_{\rm f}(t)|2\rangle\langle 2|\psi_{\rm i}(t)\rangle}{\langle\psi_{\rm f}|\psi_{\rm i}\rangle} &&= 0\ , \\ P(3,t) &= \frac{\langle\psi_{\rm f}(t)|3\rangle\langle 3|\psi_{\rm i}(t)\rangle}{\langle\psi_{\rm f}|\psi_{\rm i}\rangle} &&= \frac{p^2{\rm e}^{-\lambda_3 t_{\rm f}}}{\langle\psi_{\rm f}|\psi_{\rm i}\rangle}\ . \end{aligned} \tag{20}$$

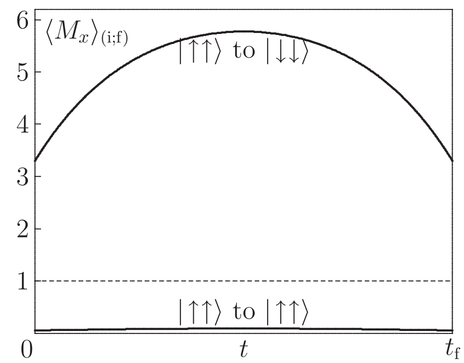

Figure 3: Abnormal and normal TTCE of M_x for $p = 0.2$ and $p^2 t_\mathrm{f} = 0.01$.

The fictitious negative probability is found in $P(1, t)$. Of course the completeness of the probability,

$$\sum_{k=0}^{3} P(k, t) = 1,$$

is satisfied evidently because of Eq.(19).

A related unusual behavior to this fact is the abnormal enhancement of some observables. For example, if we calculate the TTCE of a quantity,

$$M_x = \frac{1}{2}(\sigma_x \otimes \sigma_0 + \sigma_0 \otimes \sigma_x), \tag{21}$$

an abnormal behavior

$$\langle M_x\rangle_{(\mathrm{i;f})} = \frac{1}{\langle\psi_\mathrm{f}|\psi_\mathrm{i}\rangle}\left[\frac{2p}{1+p^2}\left(1 - p^2 \mathrm{e}^{-\lambda_3 t_\mathrm{f}}\right) - \frac{1-p^2}{1+p^2}\left(\mathrm{e}^{-\lambda_3 t} + \mathrm{e}^{-\lambda_3 (t_\mathrm{f} - t)}\right)\right] > 1,$$

is found for sufficiently small p and t_f. An example is shown in Figure.3. Note that the natural norm of M_x must be less than 1, because the eigenvalues of M_x are $\{-1, 0, 0, 1\}$. When the transition rate is very small, i.e. $p^2 t_\mathrm{f} \ll 1$, we find

$$\langle M_x\rangle_{(\mathrm{i;f})} \gg 1.$$

A plain reason of this singular behavior is that the overlap integral $\langle\psi_\mathrm{f}|\psi_\mathrm{i}\rangle$ in the denominator may be expected to be very small owing to (ii) of Eq.(14),

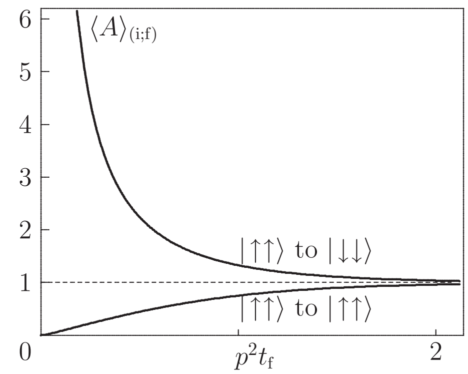

Figure 4: Abnormal and normal TTCE of $A = \sigma_x \otimes \sigma_x$ for $p = 0.2$.

whenever the initial and the final states differ from each other, i.e. $\boldsymbol{x}_{\mathrm{i}} \neq \boldsymbol{x}_{\mathrm{f}}$. This means that to reach $\boldsymbol{x}_{\mathrm{f}} = (\downarrow\downarrow)$ starting from $\boldsymbol{x}_{\mathrm{i}} = (\uparrow\uparrow)$ in a given time occurs scarcely and is far from the main flow of the conditional probability. On the contrary none of such strange behaviors are found when $\boldsymbol{x}_{\mathrm{i}} = \boldsymbol{x}_{\mathrm{f}}$, e.g. $\boldsymbol{x}_{\mathrm{i}} = \boldsymbol{x}_{\mathrm{f}} = (\uparrow\uparrow)$. The result for the latter case for the same parameters as the upper abnormal case is shown by the lower curve in Figure.3, its maximum being ~ 0.09 at $t = t_{\mathrm{f}}/2$ and minimum ~ 0.05 at $t = 0$ and t_{f}.

In Figure.4 the TTCE of another quantity $A = \sigma_x \otimes \sigma_x$ are shown also. Note that A is commutative with H and a conserved quantity. Then the horizontal axis in this figure shows a parameter of the transition probability, not the time.

5. Extension of TTCP to a density matrix

It should be noted that the physical quantities M_x and A are *non-diagonal* in the spin-state representation and have no corresponding quantities in the classical Ising spin system. They are related to the transition rate of the stochastic Ising spin. In order to calculate the expectations of such non-diagonal quantities we need an extension of the TTCP to the two-time conditional (TTC) density matrix defined by

$$\rho_{(\mathrm{i;f})}(t) \quad = \quad \frac{1}{\langle\psi_{\mathrm{f}}|\psi_{\mathrm{i}}\rangle}\,|\psi_{\mathrm{i}}(t)\rangle\langle\psi_{\mathrm{f}}(t)|$$

$$= \frac{1}{\langle\psi_{\rm f}|\psi_{\rm i}\rangle}\sum_{\boldsymbol{x},\boldsymbol{x'}}\overline{\psi}(\boldsymbol{x'},t|\boldsymbol{x}_{\rm f},t_{\rm f})\psi(\boldsymbol{x},t|\boldsymbol{x}_{\rm i},0)\ |\boldsymbol{x}\rangle\langle\boldsymbol{x'}|. \tag{22}$$

From the definition Eq.(12) of the overlap integral $\langle\psi_{\rm f}|\psi_{\rm i}\rangle$, it is evident that

$$\mathrm{Tr}\ \rho_{(\mathrm{i};\mathrm{f})}(t) = \frac{1}{\langle\psi_{\rm f}|\psi_{\rm i}\rangle}\sum_{\boldsymbol{x}}\overline{\psi}(\boldsymbol{x},t|\boldsymbol{x}_{\rm f},t_{\rm f})\psi(\boldsymbol{x},t|\boldsymbol{x}_{\rm i},0) = 1.$$

It should be noted, however, that diagonal elements of this density matrix are not always positive as is shown by Eq.(20) in Sec.4, when it is diagonalized by using the basis $\{|k\rangle, k=0,1,2,3\}$, the eigenstates of the Hamiltonian H.

With use of this density matrix the definition Eq.(15) of the TTCE is extended as

$$\langle Q\rangle_{(\mathrm{i};\mathrm{f})} = \mathrm{Tr}\ \rho_{(\mathrm{i};\mathrm{f})}Q.$$

This definition of the TTCE results in the classical one, if Q is diagonal.

The notion of this density matrix has not been used in the conventional classical stochastic process. It should be emphasized, however, that this quantity is within a scope of the classical stochastic process itself, because the wave functions, ψ and $\overline{\psi}$ in Eq.(22) are related to the forward and the posterior, classical conditional probabilities, respectively. In addition, we have an alternative expression for $\overline{\psi}$,

$$\overline{\psi}(\boldsymbol{x'},t|\boldsymbol{x}_{\rm f},t_{\rm f}) = \psi(\boldsymbol{x'},t_{\rm f}|\boldsymbol{x}_{\rm f},t)\ \left(=\phi_0(\boldsymbol{x'})^{-1}P(\boldsymbol{x'},t_{\rm f}-t|\boldsymbol{x}_{\rm f},0)\right), \tag{23}$$

or equivalently,

$$\begin{aligned}\overline{P}(\boldsymbol{x'},t|\boldsymbol{x}_{\rm f},t_{\rm f})P_0(\boldsymbol{x}_{\rm f}) &= P(\boldsymbol{x}_{\rm f},t_{\rm f}|\boldsymbol{x'},t)P_0(\boldsymbol{x'}) \\ &= P(\boldsymbol{x'},t_{\rm f}|\boldsymbol{x}_{\rm f},t)P_0(\boldsymbol{x}_{\rm f}),\end{aligned} \tag{24}$$

for $t \le t_{\rm f}$ due to the time-reversal symmetry corresponding to the detailed balance. Then the density matrix Eq.(22) can be written as

$$\rho_{(\mathrm{i};\mathrm{f})}(t) = \frac{1}{\langle\psi_{\rm f}|\psi_{\rm i}\rangle}\sum_{\boldsymbol{x},\boldsymbol{x'}}\frac{P(\boldsymbol{x'},t_{\rm f}-t|\boldsymbol{x}_{\rm f},0)P(\boldsymbol{x},t|\boldsymbol{x}_{\rm i},0)}{\phi_0(\boldsymbol{x'})\phi_0(\boldsymbol{x})}|\boldsymbol{x}\rangle\langle\boldsymbol{x'}|. \tag{25}$$

This fact means that we can define the TTC density matrix with only a pair of the usual, forward conditional probabilities for two individual initial states, $\boldsymbol{x}_{\rm i}$ and $\boldsymbol{x}_{\rm f}$.

6. Summary and discussions

Except for the facts that the time is imaginary and the wave function is always real and positive, the classical stochastic process can be described in an analogous form of the quantum mechanics, if we use the TTCP. For example, the abnormal behaviors of the weak value in the quantum mechanics are emulated. Note that the TTCP and the weak value are always real in the present classical case. Therefore, the origin of such abnormal behaviors is clearer than the quantum mechanical case where complex quantities appear. In addition, if we have not the explicit solution of the eigenvalue problem, we may calculate the weak value at least with use of the Monte-Carlo simulation which is often used to investigate the stochastic model.

The importance of the weak value in the quantum mechanics is that it is related to the new notion of the weak measurement without disturbing the quantumu state. An analogous notion of the latter in the classical stochastic process has not been found yet.

Acknowledgements

This work is supported by Open Research Center Project for Private Univercities: Matching fund subsidy from MEXT of Japan.

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